Biological Physics II - Tutorial

SS 2019
Marc Hennes
marc.hennes@uni-koeln.de
0221-470-8533

Week 10 - Turing instability

A. Wavelength of Turing Patterns

Consider the following system of differential equations which can be used to describe pattern formation treated in the lecture:

$$\dot{X} = D_x \frac{\partial^2 X}{\partial x^2} + f(X,Y)$$
$$\dot{Y} = D_y \frac{\partial^2 Y}{\partial x^2} + g(X,Y)$$

for the concentrations of morphogens X and Y, both functions of position (1D). The functions f and g reflect the chemical interactions between the morphogens. We introduce the notation:

$$a_{11} = \frac{\partial f}{\partial X}\Big|_{(h,k)}$$
$$a_{12} = \frac{\partial f}{\partial Y}\Big|_{(h,k)}$$
$$a_{21} = \frac{\partial g}{\partial X}\Big|_{(h,k)}$$
$$a_{22} = \frac{\partial g}{\partial Y}\Big|_{(h,k)}$$

where h and k are steady-state values of X and Y. We are interested in the stability of this homogeneous steady state $(X,Y) = (h,k)$.

1) Linearise the reaction-diffusion equations in the vicinity of the fixed point (h,k). Consider the time evolution of a periodic perturbation of the form $(\delta X, \delta Y) = (\eta(t)e^{iqx}, \epsilon(t)e^{iqx})$, and show that the dynamics of the perturbation is described in terms of the matrix (only keep the real parts):

$$A = \begin{pmatrix} -q^2 D_x + a_{11} & a_{12} \\ a_{21} & -q^2 D_y + a_{22} \end{pmatrix} \tag{0.1}$$

2) Show that the Eigenvalues can be written as: $\lambda = \frac{1}{2}(\text{tr}(A) \pm \sqrt{\text{tr}^2(A) - 4\text{det}(A)})$, and that the requirement for stability is that the trace is negative, and the determinant positive.

3) Show that a necessary condition for linear stability of a fixed point in the absence of diffusion is that at least one of a_{11} or a_{22} is negative.

4) Show that for a fixed point that is stable in the absence of diffusion, a necessary condition for linear *instability* in the presence of diffusion is that exactly one of a_{11} and a_{22} is negative. To this end, proceed as following:

- Calculate $\text{det}(A)$
- Then find the minimum of $\text{det}(A)$ with respect to q^2.
- Insert this q_{min} into $\text{det}(A)$
- Show that $\text{det}(A(q_{min})) < 0$ is equivalent to: $D_x a_{22} + D_y a_{11} > 2\sqrt{D_x D_y (a_{11}a_{22} - a_{12}a_{21})}$.
- Argue that $a_{11}a_{22} - a_{12}a_{21}$ is positive, and hence the square root. The assumption follows.

4) Using $D_x = a_{11}\lambda_x^2$, and $D_y = -a_{22}\lambda_y^2$, calculate q_{min}^2. To what does this wavenumber correspond?

BONUS: explain what the equation describes, and their biological context. What does instability mean?

Good luck :)