Biological Physics II - Tutorial

SS 2019
Marc Hennes
marc.hennes@uni-koeln.de
0221-470-8533

Week 1 - Introduction to numerical integration and fixed points

For the first week, we want to warm-up and refresh our knowledge regarding the numerical solutions of differential solutions that are not solvable analytically :) And treat and identify fix-points.

A: Numerical solution of an ODE

Consider the following general ODE, describing dynamical systems:

$$\frac{dy}{dt} = f(t, y)$$

Goal: Solve this equation for a given $f(t, y)$, and initial condition $y(t = 0)$, with $t > 0$.
Terminology:

- the vector $\mathbf{y}(t)$ is called the **state**
- $\mathbf{y}(0)$: initial condition / initial state
- n: dimension of the system
- $y_0, y_1, y_2, ...$: dynamical variables

Example: Lotka-Volterra (LV) model

$$f_{LV}(t, y) = (y_1(3 - y_1 - 2y_2), y_2(2 - y_2 - y_1))^T$$

1) What are the dynamical variables?
2) Solve this equation numerically using Python, Matlab, or Julia, with the initial conditions $y_1 = 1$, $y_2 = 2$. Use
a) an explicit Euler scheme for the solver:

$$y_{(1,2)}(t + \Delta t) = y_{(1,2)}(t) + \Delta t f_{LV}(t, y_{(1,2)}(t))$$

with a timestep of $\Delta t = 0.01$, and 1000 iterations (the explicit Euler solver is the first order truncation of the Taylor expansion). And
b) the fourth order Runge-Kutta method (again with $\Delta t = 0.01$, 1000 iterations total):

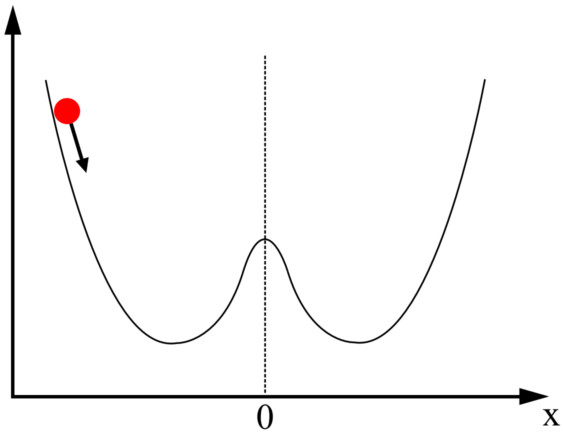
$$\begin{aligned}
k_1 &= f(t, y) \\
k_2 &= f(t + \frac{1}{2}\Delta t, y(t) + \frac{1}{2}\Delta t k_1) \\
k_3 &= f(t + \frac{1}{2}\Delta t, y(t) + \frac{1}{2}\Delta t k_2) \\
k_4 &= f(t + \Delta t, y(t) + \Delta t k_3) \\
\rightarrow y(t + \Delta t) &= y(t) + \frac{1}{6}\Delta t(k_1 + 2k_2 + 3k_3 + k_4)
\end{aligned}$$

Here, calculate first at each time iteration the four k's (calculate first k_1, then using k_1, calculate k_2, ...), then calculate $y(t + \Delta t)$.
For both methods, save the solutions $y_{(1,2)}$ at each timestep in arrays, and plot them as a function of time.
c) do both methods give the same results? Do they converge? What is the meaning behind the different terms in the LV model? Find examples where both methods would give different results.
d) BONUS: increase the timestep Δt. At what point do you get non-physical results?

Good luck! If the questions are too hard, feel free to write or come by, and i can get you started with a functioning example program in MATLAB.

B: Fixed points and phase plots

1) For the classical problem of a swinging pendulum, find the corresponding differential equation describing the system, explain the terms, and draw the phase plot (with appropriate dynamical variables) in the presence and absence of friction. Discuss the case of weak and strong friction.
2) For the case of a point particle moving in a mexican hat potential, draw the phase plot for different interesting initial points (different positions, different velocities, choose appropriate dynamical variables). Find the fixed points and describe them (stable, unstable, ...).



Mexican hat potential